Received: *November 3, 1998,* Accepted: *December 7, 1998*

Dualities and the $SL(2,Z)$ anomaly

Sunil Mukhi

Lyman Laboratory of Physics, Harvard University,
Cambridge, MA 02138, U.S.A., and
Tata Institute of Fundamental Research,
*Homi Bhabha Rd, Bombay 400 005, India**
E.mail: `mukhi@tifr.res.in`

Abstract: The $SL(2,Z)$ anomaly recently derived for type IIB supergravity in 10 dimensions is shown to be a consequence of T-duality with the type IIA string, after compactification to 2 dimensions on an 8-fold. This explains the identity of the gravitational 8-forms appearing in different contexts in the effective actions of type IIA and IIB string theories. In this framework, constraints on the compactification manifold arise from modular invariance of the 2d theory. Related issues in 6 dimensions are examined, and it is argued that similar anomalies are present on the worldvolumes of M-theory 5-branes and orientifold 5-planes.

Keywords: M-Theory, String Duality, Anomalies in Field and String Theories.

*Permanent address.

Contents

1. Introduction

It has recently been observed [1] that a certain chiral $U(1)$ transformation on fermions in 10d type IIB supergravity is anomalous. This anomaly is not irreducible, but rather can be cancelled by the addition of a local counterterm. The counterterm fails to be invariant under $SL(2,Z)$ duality transformations of the theory, hence this result is interpreted as an $SL(2,Z)$ anomaly. It turns out that the counterterm leading to this anomaly vanishes when evaluated on most "standard" compactifications of type IIB, since it depends on derivatives of the dilaton and axion fields over the compactification manifold. Hence it turns out to give constraints only on compactifications of "F-theory" [2] type.

The anomaly is proportional to a certain 8-form built out of Riemann curvature 2-forms. This turns out to be precisely the form whose integral over an 8-manifold gives the Euler characteristic of the manifold. The same 8-form appears in the low-energy Lagrangians of type IIA string theory [3] as well as M-theory [4], but has apparently not appeared so far in the Lagrangian of type IIB string theory. In particular, its appearance in the present context seems to be a remarkable coincidence.

One purpose of this note is to provide an alternate derivation of the $SL(2,Z)$ anomaly using T-duality and freedom from gravitational anomalies. Among other things, this provides an explanation for the apparent coincidence noted above.

T-duality between type IIA and IIB string theories compactified on at least one spatial circle, requires the presence of certain couplings on compactified type IIB, which, however, are absent in the uncompactified theory. These couplings lead to tadpoles in naive low-dimensional compactifications of the type IIA theory [5]. It was explained in ref. [6] that this strange phenomenon in type IIB arises from the chirality of the theory, which is preserved by its 8-fold compactifications down to 2 spacetime dimensions. This chirality induces a "vacuum momentum" (first noted in a similar context in ref. [7]) which is related by T-duality to the type IIA tadpole, or "vacuum winding charge". In ref. [6], the spectrum of compactified IIB theory was explicitly computed and shown to reproduce the expected vacuum momentum.

All this is related to the $SL(2,Z)$ anomaly as follows. Suppose we compactify the type IIB string on an 8-fold down to 2 dimensions. Because the resulting theory is chiral, the anomalous $U(1)$ of ref. [1] potentially survives compactification. As will be shown below, a linear relation holds between this $U(1)$ anomaly, the two-dimensional gravitational anomaly, and the vacuum momentum in 2d. Although potentially the 2d chiral theory has gravitational anomalies, these were shown to cancel in ref. [6]. It follows that the $U(1)$ anomaly is equated to the vacuum momentum of the theory.

This vacuum momentum is determined by T-duality with type IIA (and independently by the computation in ref. [6]) to be proportional to $\chi/24$ where χ is the Euler characteristic of the 8-fold. The result is that the $SL(2,Z)$ anomaly is completely predicted by this argument, which provides the link to type IIA and explains the ubiquitous role of the Euler 8-form.

In this picture, the constraint $\chi/24 \in Z$ is a consequence of modular invariance of the 2d theory. Moreover, it will be argued that this constraint is weakened if we allow for background fluxes, much as in type IIA- and M-theory. This in turn gives rise to a prediction that the anomaly computed in ref. [1] will be modified if the 4-form potential D^+ is nonzero.

The relationship between anomalies and vacuum momentum that is studied here in type IIB theory also makes a fascinating appearance in M-theory. This happens via the fact, which will be argued below, that M-theory 5-branes and orientifold 5-planes have analogous anomalies on their world volumes and give rise to vacuum momentum when wrapped on suitable 4-folds.

Throughout this paper, the principal focus will be on the anomaly in the chiral $U(1)$ symmetry described in ref. [1]. From a string theory point of view, it may be debated whether this anomaly really needs to be cancelled by a local counterterm as is done in ref. [1], since the global $SL(2,R)$ and local $U(1)$ symmetry relevant in that discussion seem to be more a feature of type IIB supergravity than of string theory. If we choose to cancel it then there is an $SL(2,Z)$ anomaly, but if we do not, the $U(1)$ anomaly remains. In either case, it is the surviving anomaly that is related by duality, as described below, to a number of interesting phenomena.

JHEP12(1998)006

2. Type IIB on 8-folds

The relevant results in ref. [6] for type IIB compactifications on 8-folds will first be reviewed briefly.

On compactifying type IIB on a circle, it becomes equivalent to IIA on a circle under T-duality. Thus it must possess the dimensional reductions of both the classical term $\int B \wedge dC \wedge dC$ and the one-loop term $\int B \wedge I_8$ in type IIA theory (here B is the NS-NS 2-form and C is the RR 3-form of type IIA, while $I_8(R)$ is a polynomial in the curvature, defined below). After reducing on a circle, the B-field of type IIA becomes a 1-form A which measures the winding charge with respect to that circle. Under T-duality this, in turn, becomes the Kaluza-Klein 1-form arising by reduction of the 10D metric of IIB on the circle. Thus we must look for terms in type IIB which reduce to $\int A \wedge dC \wedge dC$ and $A \wedge I_8$ in 9 dimensions.

The origin of the tree-level term is explained in ref. [6], and will not be relevant here. The one-loop term is far more subtle. It is known that in 10 dimensions there is no one-loop correction in type IIB analogous to the term $\int B \wedge I_8$ in type IIA. Moreover, one can easily convince oneself directly that there is no purely gravitational term that one can write down in 10d which reduces to $\int A \wedge I_8$ in 9d, with A being the KK gauge field. In fact, it turns out that no modification is required in 10d to the type IIB action, but as soon as one compactifies on a circle, however large, there is a radius-dependent term of the desired form in 9d.

Suppose we compactify both type IIA and IIB on the same 8-fold and then further on a circle to $0+1$ dimensions. T-dualizing along the circle maps one theory to the other. Now we have an apparent puzzle: type IIA has a 2-form tadpole in 2d [5], which will become a 1-form tadpole in 1d, and this is proportional to the Euler characteristic χ of the eightfold. However, there is no inconsistency for type IIB on the eightfold to two dimensions (for example, gravitational anomalies cancel, as was demonstrated in ref. [6]), so the inconsistency required by T-duality must arise upon compactifying one further dimension. Moreover, it must take the form of a tadpole for the KK 1-form $A = g_{12}$.

For 2d field theories on a cylinder, the generator of translations along the compact direction is $L_0 - \overline{L}_0$. Thus, a nonzero value of this operator in the vacuum implies that, from a 2d point of view, there is a nonzero momentum in the vacuum state. Under T-duality, this will turn into a nonzero winding charge of the vacuum, just what we would expect in a theory which has a 2-form tadpole in 2d. The tadpole must have the precise value $\chi/24$ (for the special case of $K3 \times K3$, a similar argument was given by Ganor [7]).

Explicit computation in the compactified type IIB theory indeed shows [6] that

$$(L_0 - \overline{L}_0)_{\text{vac}} = \frac{\chi}{24} \tag{2.1}$$

as predicted by T-duality.

Note that if the circle becomes large and we are effectively in two noncompact dimensions, this effect goes away. The reason is that the operator L_0 as conventionally defined in conformal field theory has a zero-point contribution $-\frac{1}{24}$ for a free boson only if the radius of the circle (the range of the σ coordinate) is fixed to be 2π. For a circle of radius $2\pi R$, the zero-point contribution is actually $-\frac{1}{24R}$, so that it goes away in the limit $R \to \infty$. This explains why there is no corresponding one-loop term in the effective action of type IIB theory in 2 (or 6 or 10) dimensions, and yet the prediction of T-duality with type IIA is satisfied.

3. The $SL(2,Z)$ anomaly

In what follows, we will always consider spin manifolds with at least one non-vanishing spinor.

Let us define the 8-form

$$I_8(R) = -\frac{1}{(2\pi)^4}\frac{1}{8}\left(\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2\right), \tag{3.1}$$

which has the property that

$$\int_{M^8} I_8(R) = \chi\,, \tag{3.2}$$

where χ is the Euler characteristic of the 8-manifold M_8.

We can also define another 8-form, the signature 8-form:

$$J_8(R) = -\frac{1}{(2\pi)^4}\frac{1}{180}\left(7\,\mathrm{tr}R^4 - \frac{5}{2}(\mathrm{tr}R^2)^2\right), \tag{3.3}$$

which satisfies

$$\int_{M^8} J_8(R) = \tau\,, \tag{3.4}$$

where τ is the signature of the 8-manifold M_8.

It is evident that there are precisely two independent 8-forms that one can make out of traces of products of the Riemann 2-form, these can be parametrized as $\mathrm{tr}R^4$ and $(\mathrm{tr}R^2)^2$, or as p_2 and $(p_1)^2$ (p_i are the Pontryagin classes), or as $I_8(R)$ and $J_8(R)$.

Now, on general grounds we can assume the $U(1)$ anomaly in 10d type IIB supergravity to be given by

$$\Delta = -\int \frac{F}{4\pi}\wedge A_8(R)\,\Sigma(x)\,. \tag{3.5}$$

Here, $\Sigma(x)$ is the parameter of the $U(1)$ transformation, and $A_8(R)$ is an unknown 8-form which can, of course, be parametrized as a linear combination of $I_8(R)$ and $J_8(R)$ that we have defined above. The 2-form F is defined in terms of the complex dilaton-axion field τ as

$$F = i\frac{d\overline{\tau}\wedge d\tau}{4(\tau_2)^2}\,. \tag{3.6}$$

The form of eq. 3.5 follows from the fact that (i) there is no nonzero 10-form made entirely from traces of products of R, (ii) the 2-form F satisfies $F \wedge F = 0$ by virtue of its definition.

With this expression for the $U(1)$ anomaly, there is a local counterterm

$$\int \phi \, \frac{F}{4\pi} \wedge A_8(R) \,, \tag{3.7}$$

where ϕ is a scalar field which is pure gauge, corresponding to the $U(1)$ part of the $SL(2,R)$ variables. Under the gauge transformation $\delta\phi = \Sigma(x)$, the variation of eq. 3.7 cancels the anomaly in eq. 3.5.

In ref. [1], it is shown by explicit computation that

$$A_8(R) = \frac{1}{6} I_8(R) \,. \tag{3.8}$$

In the following, it will be shown that this is a consequence of the vacuum momentum predicted by T-duality with the type IIA string, and confirmed in ref. [6].

4. The $U(1)$ anomaly and vacuum momentum

Consider compactifying the type IIB string on a (spin) 8-fold M^8 to 2 spacetime dimensions. The resulting 2-dimensional theory is chiral, in fact it has $(0,2)$ chiral supersymmetry for 8-folds with $spin(7)$ holonomy, $(0,4)$ for $SU(4)$ holonomy, $(0,8)$ for $SU(2) \times SU(2)$ holonomy (the case of $K3 \times K3$) and $(0,16)$ if the holonomy is contained in $SU(2)$, which is true for the orbifold T^8/Z_2. The spectrum for all these cases is worked out in ref. [6], but we will not need this for the following argument.

A generic 2d supergravity theory with $(0,N)$ supersymmetry coupled to matter has the following spectrum:

$$\begin{array}{rl} 1 \text{ supergravity multiplet}: & (g_{\mu\nu}, \phi, N\psi^-_\mu, N\psi^+) \,, \\ \binom{n_+}{N} \text{ chiral multiplets}: & (N\phi^+, N\psi^+) \,, \\ n^\phi_- \text{ anti}-\text{chiral scalars}: & \phi^- \,, \\ n^\psi_- \text{ anti}-\text{chiral fermions}: & \psi^- \,. \end{array} \tag{4.1}$$

Note that the anti-chiral matter fields are supersymmetry singlets.

Now, purely in terms of the integers $N, n_+, n^\phi_-, n^\psi_-$, we can compute various interesting quantities in this theory. For now, we are not using the fact that this 2d supergravity theory comes from type IIB or any other compactification.

Suppose this 2d theory has a chiral $U(1)$ symmetry under which the gravitinos have charge α and the other fermions have charge β. The anomaly in this symmetry can be written:

$$\Delta = -4A^{\alpha,\beta}_{U(1)} \int F \, \Sigma(x) \,, \tag{4.2}$$

where F is the $U(1)$ field strength. (In the present case the $U(1)$ gauge field is composite, and F is as given in eq. 3.6.) The coefficient of the anomaly is given by [8]:

$$A^{\alpha,\beta}_{U(1)} = \alpha N + \beta \frac{1}{24}(n_+ - n^{\psi}_-)\,. \tag{4.3}$$

This theory could also have a gravitational anomaly, for which the coefficient would be proportional to

$$A_{\rm grav} = \frac{1}{2}N + \frac{n_+ - n^{\phi}_-}{24} + \frac{n_+ - n^{\psi}_-}{48}\,. \tag{4.4}$$

Finally, the chiral theory could in principle have a non-vanishing zero-point energy. Supersymmetry actually sets this to zero for chiral fields, which lie in supermultiplets, but does not determine it for the anti-chiral fields which are supersymmetry singlets. We have:

$$A_{\rm vac} \equiv (L_0 - \overline{L}_0)_{\rm vac} = \frac{n^{\phi}_- - n^{\psi}_-}{24}\,. \tag{4.5}$$

We can now specialize to the case where the chiral $U(1)$ has charges $\frac{1}{2}, \frac{3}{2}$ on the gravitino and spin-$\frac{1}{2}$ fermions respectively. From the above three equations, we easily read off the relation

$$A^{\frac{1}{2},\frac{3}{2}}_{U(1)} = A_{\rm grav} + A_{\rm vac}\,. \tag{4.6}$$

It follows that if the 2d theory is free of gravitational anomalies, then the $U(1)$ anomaly (which leads to an $SL(2,Z)$ anomaly by the arguments of ref. [1]) is numerically equal to the vacuum momentum.

As a check, we use formulae from ref. [6] for 8-fold compactifications of the type IIB string, and find

$$A^{\frac{1}{2},\frac{3}{2}}_{U(1)} = \tfrac{\chi}{24}\,, \qquad A_{\rm grav} = 0\,, \qquad A_{\rm vac} = \tfrac{\chi}{24}\,. \tag{4.7}$$

To summarize, we have shown that vacuum momentum, itself a consequence of T-duality with the type IIA string, is the origin of the $U(1)$ anomaly in 2d. Now, because there are only two independent gravitational 8-forms, the corresponding $U(1)$ anomaly in 10d is also uniquely determined to be the one in eqs.3.5, 3.8, as found in ref. [1].

5. Constraints on compactifications

In ref. [1], the presence of a $U(1)$ anomaly leads to a counterterm which in turn induces an $SL(2,Z)$ anomaly. The consistency requirements discussed in ref. [1] imply that $\chi/24 \in Z$. As we will now see, the same constraint arises very naturally in the present framework by requiring modular invariance of the 2d theory.

Viewed as a chiral 2-dimensional CFT, the theory obtained by compactifying the type IIB string on an 8-fold has a partition function:

$$Z = \mathrm{tr}\, q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\,. \tag{5.1}$$

Under a modular transformation $\tau \to \tau + 1$, the partition function is invariant only if every state $|\Phi\rangle$ in the Hilbert space of the theory satisfies

$$e^{2\pi i\big((L_0-\overline{L}_0)-(c-\overline{c})/24\big)}|\Phi\rangle = |\Phi\rangle\,. \tag{5.2}$$

Thus the exponent is required to be an integer. In particular, the vacuum state should satisfy this requirement, from which we get:

$$\Big((L_0-\overline{L}_0)_{\rm vac} - (c-\overline{c})/24\Big) \in Z\,. \tag{5.3}$$

The left hand side is precisely the sum of the vacuum momentum and the gravitational anomaly, $A_{\rm vac} + A_{\rm grav}$. In 8-fold compactifications of type IIB, this quantity turns out to be $\frac{\chi}{24}$, as we have seen, so quantization of this number follows very naturally from modular invariance in 2d.

It should be noted that the constraint $\chi/24 \in Z$ only applies to an elementary class of compactifications *without fluxes*. It is known, for example, that in M-theory and type IIA string theory, supersymmetric compactifications on 8-folds can include nonzero values of $\int dC\wedge dC$ over the 8-fold [9]. Later it was shown [10] that dC is actually allowed to have half-integral flux over a 4-cycle, and that consistent compactifications can be defined on 8-manifolds for which χ is only a multiple of 6, but not of 24, so long as there is a suitable nonzero value of $\int dC\wedge dC$ in the background.

It has been analogously argued that various other field strengths can have fractional flux in string theory [11], but a consistent compactification of type IIB on 8-folds with non-integer $\chi/24$ does not appear to have been constructed. For F-theory, fluxes are irrelevant to this particular question, because elliptically fibred Calabi-Yau complex 4-folds anyway have integer $\chi/24$ [5].

Starting from the fact that fractional 4-form field-strengths dC must be turned on in M-theory and type IIA, for manifolds not satisfying $\chi/24 \in Z$, it will now be argued that the formulae for the $U(1)$ and $SL(2,Z)$ anomalies in ref. [1] necessarily receive corrections when the 5-form field strength dD^+ of type IIB is nonzero (here D^+ stands for the self-dual 4-form potential). However, these corrections cannot apparently be written in Lorentz-covariant form, which is a manifestation of the well-known impossibility of writing a covariant action when D^+ is nonzero.

The argument goes as follows. Suppose we pick M^8 to be a Calabi-Yau complex 4-fold with χ a multiple of 6 but not of 24. Now compactify type IIA theory on this down to 2 dimensions, and include a half-integral flux dC over 4-cycles so that

$$\frac{\chi}{24} - \frac{1}{8\pi^2}\int dC\wedge dC \in Z\,. \tag{5.4}$$

If this quantity is moreover positive, it can be cancelled by including the right number of type IIA strings in the vacuum and we have a tadpole-free compactification (the relevant equation, correcting a sign in [9] and incorporating the vacuum branes of ref. [5], can be found in ref. [6]). But for the moment we do not include these branes, and just consider this compactification to 2d, with its associated tadpole problem.

Compactifying further on a circle and T-dualizing implies that the T-dual type IIB has a vacuum momentum

$$(L_0 - \overline{L}_0)_{\rm vac} = \frac{\chi}{24} - \frac{1}{8\pi^2}\int dC \wedge dC \in Z\,. \tag{5.5}$$

What is dC from the type IIB point of view? If we label the circle direction as x^9 and let the coordinates of the 8-fold be x^i, $i = 1,\ldots,8$ then we have the T-duality relation:

$$C_{ijk} = D^+_{ijk1}\,, \tag{5.6}$$

from which it follows that

$$\begin{aligned}\int_{M^8} dC \wedge dC &\sim \int_{M^8} \epsilon^{ijkli'j'k'l'}(dD^+)_{ijkl1} \wedge (dD^+)_{i'j'k'l'1} \\ &\sim (dD^+)_{ijkl0}(dD^+)^{ijkl}{}_{1}\,,\end{aligned} \tag{5.7}$$

where the last equality follows from self-duality of dD^+ in 10 dimensions. This is of course not manifestly covariant under $SO(9,1)$, but only under the subgroup $SO(8) \times SO(1,1)$.

Tracing back the relationship discussed in the previous sections, we find that the formula for the $U(1)$ anomaly in 10 dimensions must be modified as follows:

$$\Delta = -\int \frac{F}{4\pi} \wedge \frac{I_8(R)}{6}\,\Sigma(x) \to -\int \frac{F}{4\pi} \wedge 4\left(\frac{I_8(R)}{24} - \frac{1}{8\pi^2}(dD^+)_1 \wedge (dD^+)_1\right)\Sigma(x), \tag{5.8}$$

where the extra term is shorthand for the last term in eq. 5.7. Perhaps it is possible to write this is in a better way, as in ref. [12], but that direction will not be pursued here. The main point is to note that the $SL(2,Z)$ anomaly does not necessarily imply integer quantization of $\chi/24$, but that background fluxes will provide an "escape route" from this rule, as in type IIA and M-theory.

6. Relation to M-theory 5-branes and orientifold 5-planes

In this section we note that the above considerations have some implications for 5-branes and orientifold 5-planes of M-theory.

Suppose we compactify type IIB string theory to 6 dimensions on K3. The 10-dimensional $U(1)$ anomaly of ref. [1] descends to a 6-dimensional $U(1)$ anomaly by simply using $p_1(K3) = 48$, from which one finds:

$$\Delta = 2 \int \frac{F}{4\pi} \wedge p_1(R)\, \Sigma(x)\,. \tag{6.1}$$

Clearly, a further compactifiction on $K3$ will reproduce the results above, for the special case where the 8-fold is $K3 \times K3$. Instead of doing that, we first use a nontrivial duality that relates the present model to the orientifold of M-theory on T^5/Z_2 [13, 14]. This compactification has 16 M-theory 5-branes and 32 orientifold 5-planes in the vacuum. These are the only chiral objects in the theory; the bulk is 11-dimensional M-theory and hence non-chiral.

Now we ask what is the interpretation of the above anomaly in terms of the M-theory defects. Because the bulk is non-chiral, one must assume that the anomaly comes from branes and/or planes. It turns out that this involves an interesting phenomenon about M-branes and planes[1].

It is well-known that D-branes in type II string theories carry WZ-type gravitational couplings on their world-volumes [16, 17], and more recently it has been noted that orientifold planes in the same theories also carry localized gravitational WZ couplings [11, 18]. However, the question of existence of gravitational couplings on defects in M-theory has not so far been answered, though it was raised at the end of a recent paper [19].

Actually it is quite easy to see that the M-theory 5-brane indeed cannot have gravitational WZ couplings on its world-volume (such couplings would have to be proportional to the 4-form $p_1(R)$ and another spacetime 2-form, but the latter does not exist in uncompactified M-theory). However, once we compactify M-theory on a circle and wrap the 5-brane on this circle, we obtain the D 4-brane of type IIA theory and this certainly has a gravitational coupling $\int A \wedge p_1(R)$ on its world-volume, where A is the Ramond-Ramond 1-form of type IIA theory or the Kaluza-Klein gauge field of compactified M-theory. Thus we have a puzzle: how does the M 5-brane produce this term when wrapped on a circle given that it does not have it to start with? The same question can also be asked about the orientifold 5-plane, since after wrapping, it is dual to the orientifold 4-plane of type IIA, which again has gravitational WZ couplings [11].

The question is quite analogous to the one asked in ref. [6] about the origin of $\int A \wedge I_8(R)$ in type IIB theory compactified on a circle to 9d, given that no corresponding term is present in 10d. The answer also turns out to be analogous, and fits beautifully with the $SL(2,Z)$ anomaly in eq. 6.1 above.

The key point is that, like the type IIB theory in 10d, the M 5-brane and M 5-plane are chiral objects. Hence, when they wrap on a suitable 4-fold, the resulting 2d theories are still chiral, indeed if the 4-fold is $K3$ then one gets theories with chiral

[1] Related issues were discussed in ref. [15]. Some of the observations below emerged in subsequent discussions with K.S. Narain.

supersymmetry in 2d. These theories can possess a vacuum momentum, which after compactification of one more dimension, turns into the expected tadpole predicted by the relation to D-branes and string theory orientifold planes.

This in turn means that the 6-dimensional field theories on the 5-brane and orientifold 5-plane have $U(1)$ anomalies. In fact, in this case one expects a multiplet of anomalies, since the duality group is $SO(5,21,Z)$, much larger than $SL(2,Z)$ of ten-dimensional type IIB. The details need not be spelled out here since they follow in a straightforward way from analogous considerations elsewhere in this paper and in ref. [11]. The result is that the $U(1)$ under which gravitinos have charge $\frac{1}{2}$ and the remaining fermions have charge $\frac{3}{2}$ is anomalous on an M-theory 5-brane, the anomaly being

$$\Delta = \frac{1}{12}\int_{M^6} \frac{F}{4\pi} \wedge p_1(R)\,\Sigma(x)\,. \tag{6.2}$$

In other words, M 5-branes each contribute a fraction $\frac{1}{24}$ of the total anomaly in eq. 6.1 above.

Similarly, this $U(1)$ is anomalous on an M-theory orientifold 5-plane, the anomaly in this case being

$$\Delta = \frac{1}{48}\int_{M^6} \frac{F}{4\pi} \wedge p_1(R)\,\Sigma(x) \tag{6.3}$$

or a fraction $\frac{1}{96}$ of the total. The above two formulae added up over 16 M 5-branes and 32 M orientifold 5-planes precisely reproduce eq. 6.1. Moreover, the M-theory relation with type IIA is satisfied, with these anomalies being related to vacuum momentum and hence eventually to gravitational WZ couplings.

One might worry that in this case the relationship between $U(1)$ anomalies and vacuum momentum is not so obvious, since the M 5-brane and orientifold 5-planes apparently have gravitational anomalies. However, as stressed in ref. [14], gravitational anomaly inflow from the bulk actually renders both of these objects anomaly-free separately.

The principal consequence of the above discussion is that the $U(1)$ anomaly of ref. [1] is manifested in M-theory, but not in the bulk (this would be impossible just because the bulk is non-chiral). It appears on the chiral objects of the theory, namely 5-branes and orientifold 5-planes.

7. Conclusions

We have shown that the $U(1)$ and $SL(2,Z)$ anomalies recently discussed in ref. [1] are consistently connected to a number of stringy dualities and even to M-theory. This is particularly satisfying since the original derivation is based more directly on properties of supergravity than of string theory.

We have re-derived this anomaly using the fact that type IIB string theory has a hidden "vacuum momentum" which in turn is predicted by T-duality with type IIA theory. The relation of this vacuum momentum to the $U(1)$ anomaly is embodied in eq. 4.6 above, along with the absence of gravitational anomalies in 8-fold compactifications.

It would be even more satisfying to have a direct derivation of eq. 4.6 on general grounds. Each term is a kind of anomaly in a different symmetry: $A_{U(1)}$ is of course the $U(1)$ anomaly, $A_{\rm grav}$ is the gravitational anomaly and $A_{\rm vac}$, the vacuum momentum crucial to this story, is roughly like an anomaly in the $U(1)$ rotation generated by $L_0 - \overline{L}_0$.

The constraint that $\chi/24 \in Z$ turns out to come from modular invariance in this context. It can be modified by the inclusion of background flux, which leads to the prediction that the $U(1)$ anomaly of ref. [1] should be modified if the self-dual 4-form is turned on.

Finally, it was observed that the M5-brane and M orientifold 5-plane have vacuum momentum hidden in them by virtue of their chirality, this fits in with nontrivial M-theory dualities and also with their direct relation to 4-branes and 4-planes of type IIA string theory. In turn, this phenomenon is related to $U(1)$ anomalies on the world-volumes of these objects, similar to those found in 10 dimensional type IIB theory in ref. [1].

It may be hoped that these various facts suggest something about the larger picture in which they fit together. For example, it is intriguing that none of the M-theory defects (branes and planes) have gravitational couplings of the type carried by D-branes (membranes, like D2-branes, have too low a dimension to support gravitational Chern-Simons-type couplings). The discussions above also illustrate once more the general theme that branes and planes are similar in some respects, the principal difference being that the latter have no independent world-volume fields of their own.

Acknowledgments

I am grateful to Rajesh Gopakumar, Albion Lawrence and Samir Mathur for helpful discussions, and Ruben Minasian for a useful correspondence. Hospitality and financial support from Andy Strominger and the Physics Department at Harvard University are gratefully acknowledged.

References

[1] M. Gaberdiel and M.B. Green, *An $SL(2,Z)$ anomaly in IIB supergravity and its F-theory interpretation*, `hep-th/9810153`.

[2] C. Vafa, *Evidence for F-theory*, *Nucl. Phys.* **B 469** (1996) 403 [`hep-th/9602022`].

[3] C. Vafa and E. Witten, *A one loop test of string duality*, *Nucl. Phys.* **B 447** (1995) 261 [`hep-th/9505053`].

[4] M.J. Duff, J.T. Liu and R. Minasian, *Eleven-dimensional origin of string/string duality: a one loop test*, *Nucl. Phys.* **B 452** (1995) 2610 [hep-th/9506126].

[5] S. Sethi, C. Vafa and E. Witten, *Constraints on low-dimensional string compactifications*, *Nucl. Phys.* **B 480** (1996) 213 [hep-th/9606122].

[6] K. Dasgupta and S. Mukhi, *A note on low-dimensional string compactification*, *Phys. Lett.* **B 398** (1997) 285 [hep-th/9612188].

[7] O. Ganor, *Compactification of tensionless string theories*, hep-th/9607092.

[8] L. Alvarez-Gaumé and E. Witten, *Gravitational anomalies*, *Nucl. Phys.* **B 234** (1984) 269.

[9] K. Becker and M. Becker, *M-theory on eight manifolds*, *Nucl. Phys.* **B 477** (1996) 155 [hep-th/9605053].

[10] E. Witten, *On flux quantization in M-theory and the effective action*, *J. Geom. Phys.* **22** (1997) 1 [hep-th/9609122].

[11] K. Dasgupta, D.P. Jatkar and S. Mukhi, *Gravitational couplings and Z_2 orientifolds*, *Nucl. Phys.* **B 523** (1998) 465 [hep-th/9707224].

[12] M. Henneaux and C. Teitelboim, *Dynamics of chiral (selfdual) p-forms*, *Phys. Lett.* **B 206** (1988) 650.

[13] K. Dasgupta and S. Mukhi, *Orbifolds of M-theory*, *Nucl. Phys.* **B 465** (1996) 399 [hep-th/9512196].

[14] E. Witten, *Five branes and M-theory on an orbifold*, *Nucl. Phys.* **B 463** (1996) 383 [hep-th/9512219].

[15] I. Antoniadis, S. Ferrara, R. Minasian and K.S. Narain, *R^4 Couplings in M and type II theories on Calabi-Yau spaces*, *Nucl. Phys.* **B 507** (1997) 571 [hep-th/9707013].

[16] M. Bershadsky, V. Sadov and C. Vafa, *D-branes and topological field theories*, *Nucl. Phys.* **B 463** (1996) 420 [hep-th/9611222].

[17] M.B. Green, J.A. Harvey and G. Moore, *I-brane inflow and anomalous couplings on D-branes*, *Class. and Quant. Grav.* **14** (1997) 47 [hep-th/9605033].

[18] K. Dasgupta and S. Mukhi, *Anomaly inflow on orientifold planes*, *J. High Energy Phys.* **03** (1998) 004 [hep-th/9709219].

[19] B. Craps and F. Roose, *Anomalous D-brane and orientifold couplings from the boundary state*, hep-th/9808074.